Agency in Mathematical Practice

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Abstract

A characteristic feature of the philosophy of mathematical practice is to attend to what people do when they do mathematics. But what does it mean to *do* mathematics? This question raises several issues regarding the nature of action, activity, and agency in mathematical practice. The present chapter reviews contributions in the field that have attempted to theorize about these notions. It begins with some motivations for taking agents seriously in the philosophical study of mathematical practice. The core of the chapter discusses, in turn, what it means to carrying out mathematical activities, doing things with mathematical artefacts, engaging with mathematical proofs, and performing mathematical actions prescribed by mathematical texts. Taken together, the various lines of work reported here provide an initial, but already sophisticated, picture of what it means to do mathematics. The chapter ends with some suggestions for future research on agency in mathematical practice.

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1 Introduction

Mathematics is something that people do, from the research mathematician trying to prove Goldbach's conjecture to the student in elementary school working on her mathematics homework. But what does it mean to *do* mathematics? One may expect that philosophical accounts of the nature of mathematics will provide some clues on this question. And yet, as Kenneth Manders remarked: "[c]urrent philosophical conceptions of mathematics are predominantly agentless" (Manders 2008, p. 118). Paradigmatic examples are the main developments in the foundations of mathematics where the objects of study are the notions of mathematical theories, proofs and computations. The branches of mathematical logic investigating these notions—model theory, proof theory, and recursion theory—do not require any appeal to agents doing things. The same observation can be made for most philosophical theorizing on the nature of mathematical objects and mathematical truths. The situation seems different for epistemological considerations aiming to explain how we can possess mathematical knowledge, since knowledge needs to be attributed to an epistemic subject.[1] But here again, it is easy to set aside considerations on actions, activities and agency and to only focus on the conditions that need to be met to attribute knowledge to an epistemic subject, without describing what a subject needs to do to acquire this knowledge in the first place.

One way the philosophy of mathematical practice[2] departs from traditional approaches in the philosophy of mathematics is precisely by taking into consideration the *active* dimension of mathematics, i.e., by attending to what people *do* when they do mathematics. This has become a leitmotif in describing the field, for instance:

> The focus has turned thus to a consideration of what mathematicians are actually doing when they produce mathematics. [...] These historians and philosophers agree that there is more to understanding mathematics than a study of its logical structure and put much emphasis on mathematical activity as a human activity. (Mancosu, Jørgensen, and Pedersen 2005, p. 1)

> The philosophy of mathematics has experienced a renewal in recent years due to a more open and interdisciplinary way of asking and answering questions. Traditional philosophical concerns about the nature of mathematical objects and the epistemology of mathematics are combined and fructified with the study of a wide variety of issues about the way mathematics is done, evaluated, and applied [...]. (Ferreirós 2016, p. xi)

Once it has been acknowledged that the active dimension of mathematics is a proper object of philosophical inquiry, the next step is to figure out how to study it. This raises methodological issues on how to investigate the whole range of actions and activities present in mathematical practices as well as the different forms of agency involved. It also raises theoretical questions regarding the nature of actions, activities, and agency in the specific context of mathematical practice.

The aim of this chapter is to review various ways in which the notions of action, activity, and agent have played a role in philosophical studies of mathematical practice. The focus will be on contributions that have attempted to theorize about these notions. We begin, in section 2, with some motivations for taking agents seriously when investigating mathematical practice. Section 3 is concerned with the notion of mathematical activity, both as an object of study in itself and as a way of building an epistemology of mathematics where mathematical knowledge is rooted into elementary mathematical activities. In sections 4 and 5, we will see how the notions

1. On the conceptualization of epistemic subjects in the context of mathematics, see the chapter by De Toffoli in this Handbook (De Toffoli 2023).

2. For recent overviews of the field, see Carter (2019) and Hamami and Morris (2020).

of action and agent have played an important role in the philosophical study of mathematical artefacts and mathematical proofs. Section 6 will be concerned with mathematical agency and mathematical texts, and more specifically with how mathematical actions can be prescribed by mathematical texts. Taken together, these contributions already provide important clues on what it means to do mathematics. Section 7 concludes the chapter with some suggestions on how to pursue further the study of agency in mathematical practice.

2 Why taking agents seriously

An obvious reason to take agents seriously is that virtually all the mathematical practices one may be interested to investigate would not exist without the presence of a number of agents engaging in it at a given point in human history. As Ferreirós emphasized: "[m]athematical practice always involves communities of agents in interaction" and "there is no practice without practitioners" (Ferreirós 2016, p. 45).[3] One may then expect that the notion of agent will, one way or another, figures in any philosophical analysis of mathematical practice.

Still, one may object that the essence of a mathematical practice lies in the norms or rules that are constitutive of it, and not in its particular, contingent implementation(s). From this perspective, one could investigate a given mathematical practice independently of its implementation(s), in the same way as one could study a judiciary system or an algorithm which may never have been implemented concretely. In opposition to this view, Manders (2008) has argued that, in the context of his analysis of diagram-based geometric reasoning in ancient Greece, taking into account different *roles* for the agents engaged in this practice was important in explaining its *success*. Manders distinguishes two roles that he considers of theoretical importance (see Manders 2008, p. 122): that of *asserting* the different geometrical claims made in the text and taking responsibility for them, and that of providing critical scrutiny or *probing*. One way of probing a geometrical claim is to propose a *case*, that is, a diagram which satisfies the stipulations of the claim but which is topologically different from the diagram(s) considered by the protagonist. According to Manders, proposing cases is theoretically significant because it is an essential strategy to cope with the open-ended nature of case-branching in diagram-based geometric practice, i.e., with the fact that such a practice does no possess a procedure to make sure that all topologically different diagrams that are relevant in a geometric proof have been considered. As evidence for the importance of the role of probing, Manders argues that several propositions in Euclid's *Elements* can be conceived as responses to probing claims that could potentially be raised with respect to other geometrical proofs.

Another reason to take agents seriously is when the mathematics themselves are primarily about actions. Typical examples are mathematical practices geared towards geometric constructions or concrete arithmetical/algorithmic procedures. For instance, ancient Greek geometry has been described by Reviel Netz as "the study of spatial action, not of visual representation" (Netz 1999, p. 60). Some mathematical traditions, such as those from Mesopotamia or ancient China, are primarily concerned with procedures for computations which are, by definition, set of actions to be performed (see, e.g., Chemla 2012). When analyzing mathematical practices of this kind, one may naturally be led to reflect on the nature of the actions involved and on the structure of the agents performing them.[4]

But one of the main reasons to take agents seriously is that the notions of action, activity, and agent have revealed themselves theoretically useful and fruitful when addressing a number of issues that appear prominently in the research agenda of the philosophy of mathematical

3. On Ferreirós' view of what a mathematical practice is, see also Ferreirós' chapter in this Handbook (Ferreirós 2022).

4. We can also mention here practices that could be qualified as mathematical such as string figure-making (see, e.g., Vandendriessche 2015) or paper folding (see, e.g., Friedman 2018), though of course whether such practices should be qualified as mathematical is a debated question.

practice. As we shall see in the following sections, this bundle of notions occupies an important place in philosophical work on mathematical activities, mathematical artefacts, mathematical proofs, and mathematical texts.

3 Agency and mathematical activities

Focusing on what it means to do mathematics amounts to see mathematics as an activity. In this section, we will be concerned with two lines of research that naturally emerge from this perspective. The first one consists in developing an epistemology of mathematical activities. The second one explores the possibility that mathematical knowledge is rooted or grounded into elementary human activities.

In a contribution entitled "Mathematical activity", Giaquinto (2005) started from the observation that, although philosophers of mathematical practice have set for themselves the task of investigating the activities that mathematicians engage in, they have so far focused on a narrow set of activities. Giaquinto proposed then a "preliminary map" of mathematical activities worthy of philosophical investigations, recognizing that this list is not meant to be exhaustive. His list consists of the following mathematical activities: discovery; explanation; formulation; application; justification; representation. Once such a list is established, numerous questions arise: What is the nature of each of these activities? Are they really different kinds of activities? What are the relations between them? Can they be decomposed into smaller activities? Giaquinto tackled some of these questions. In particular, he argues that discovering, explaining, and justifying are distinct kinds of activities which do not reduce to proving. He also proposes that, for each of the activities in the list, one can distinguish between making it, presenting it, and taking it in. For instance, in the case of discovering, one can distinguish between making the discovery, presenting the discovery to other people, and being the one who is trying to "get" the discovery, that is, who is receiving or taking in the discovery. Giaquinto conceives of his contribution as a "springboard" to pursue further philosophical investigations into the nature of mathematical activities.

Some authors have argued that the origins, grounds, or roots of mathematical knowledge are to be found in certain elementary human activities. The mathematician Saunders Mac Lane put the general idea as follows:[5]

> [M]athematics started from various human activities which suggest objects and operations (addition, multiplication, comparison of size) and thus lead to concepts (prime number, transformation) which are then embedded in formal axiomatic systems (Peano arithmetic, Euclidean geometry, the real number system, field theory, etc.). These systems turn out to codify deeper and nonobvious properties of the various originating human activities. (Mac Lane 1981, p. 463)

Mac Lane offered the following list of correspondence between elementary human activities and domains of mathematics:

Counting	:	to arithmetic and number theory;
Measuring	:	to real numbers, calculus, analysis;
Shaping	:	to geometry, topology;
Forming (as in architecture)	:	to symmetry, group theory;
Estimating	:	to probability, measure theory, statistics;
Moving	:	to mechanics, calculus, dynamics;
Calculating	:	to algebra, numerical analysis;
Proving	:	to logic;
Puzzling	:	to combinatorics, number theory;
Grouping	:	to set theory, combinatorics.

5. See also Mac Lane (1986, chapter XII).

There are different ways in which this general idea can be exploited to articulate a view of mathematics and mathematical knowledge as rooted or grounded into elementary human activities. I will now describe the proposals of Kitcher (1984) and Ferreirós (2016).

In chapter 6 of *The Nature of Mathematical Knowledge*, Kitcher (1984) endeavors to develop an account of mathematical truth according to which mathematical statements are true in virtue of certain operations or manipulations humans can perform in the world. In the case of arithmetic, two such operations are collecting and correlating: collecting is the activity of segregating or putting objects together; correlating is the activity of matching, relating, or connecting objects. According to Kitcher, the first type of operation leads to notions like set, number, and arithmetical operations, while the second type leads to notions like that of a function. Thinking in terms of mathematical activities such as collecting and correlating is key to Kitcher's mathematical ontology: "One central idea of my proposal is to replace the notions of abstract mathematical objects, notions like that of a collec*tion*, with the notion of a kind of mathematical activity, collec*ting*" (Kitcher 1984, p. 110). Kitcher emphasizes that the operations or manipulations he has mind are not those of any given human agent, but that of an ideal agent. This ideal agent should be thought as an idealized version of human agents. Kitcher conceives then of arithmetic as an "idealizing theory": "the relation between arithmetic and the actual operations of human agents parallels that between the laws of ideal gases and the actual gases which exist in our world" (Kitcher 1984, p. 109). This approach allows Kitcher to argue that arithmetic constitutes a description of the structure of reality, that is, of some operations that are possible to do in the physical world. In a slogan, Kitcher says that arithmetic is true: "in virtue not of what *we can do* to the world but rather of what *the world* will let us do *to it*" (Kitcher 1984, p. 108). A main goal of *The Nature of Mathematical Knowledge* is then to show how advanced mathematics can emerge from such proto-mathematical knowledge through various stages of rational transitions.

Like Kitcher, Ferreirós holds that an account of mathematical knowledge should attribute a key role to certain elementary activities or practices: "I shall argue that our knowledge of mathematics cannot be understood without emphasizing the practical roots of math, including its roots in scientific practices and technical practices" (Ferreirós 2016, p. 5). But contra Kitcher, Ferreirós' approach is not reductionist—central to Ferreirós' account is the "interplay" or "interactions" between practices, not the transitions between different mathematical practices. Another important difference with Kitcher is that Ferreirós emphasizes the cognitive dimension of elementary mathematical practices and the fact that the cognitive abilities involved must be reenacted for an individual to gain mathematical knowledge, while Kitcher situates elementary mathematical practices at the historical roots of his genealogical account. Ferreirós also offers a precise definition of what he calls a "technical practice", namely: "a recognizable type of activity that is done—and can be taught and learned—by human agents, involving direct manipulation of objects in the world, through the use of human-made instruments" (Ferreirós 2016, pp. 40–41). The three technical practices discussed by Ferreirós (2016) are that of counting, measuring, and drawing geometrical forms. An important thesis in Ferreirós' view is that knowledge of mathematics requires the mastering of these elementary techniques.

The notion of mathematical activity, intimately connected to that of a mathematical practice, is bound to play an important role in the philosophy and history of mathematical practice. As a matter of fact, many studies in the field precisely consist in the analysis of mathematical activities. The focus in this section has been on philosophical developments that have attempted to theorize about the notion of mathematical activity itself and/or to use it in the pursuit of larger goals. Here, much is to be gained by pursuing further the inquiry initiated by Giaquinto (2005). In particular, it would be theoretically and practically useful to identify and characterize the main components of what constitute mathematical activities as well as the mathematical agents able to perform them.[6] This could not only yield methodological tools to investigate

6. As an example, Carter (2008) argues that investigating the activities that mathematicians perform with

specific mathematical activities, it could also provide a perspective to compare and relate different mathematical activities—a theme central to approaches like the one of Ferreirós which emphasize the interplay and interactions between mathematical practices, and thus between mathematical activities. Possessing a rich and precise account of the nature of mathematical activities will, in turn, be directly useful for building philosophical conceptions of mathematics rooted in mathematical activities.

4 Agency and mathematical artefacts

Doing mathematics involves doing things with mathematical artefacts such as diagrams, symbols, and graphs. A significant part of the literature in the philosophy and history of mathematical practice is dedicated to understanding and reconstructing what agents did or do with mathematical artefacts in various mathematical practices, from the use of geometric diagrams in Ancient Greece mathematics (e.g., Netz 1999; Manders 2008) to that of commutative diagrams in contemporary homological algebra (e.g., De Toffoli 2017). The aim of this section is not to review this literature, which would simply be an impossible task. Rather, the focus will be on contributions that have attempted to conceptualize and theorize about what it means to do things with mathematical artefacts. This line has been pursued mainly in the cases of mathematical diagrams and mathematical symbols, and I will review here some representative studies in this trend. Before that, I will present the notion of *epistemic action* from Kirsh and Maglio (1994) which has played a central role in these discussions.

The distinction between pragmatic and epistemic actions was introduced by Kirsh and Maglio (1994) in their psychological study of the video game Tetris. Pragmatic actions are physical actions performed in the world whose aim is to bring a physical system closer to a specific goal. In Tetris, the pragmatic actions are those aiming to orient the falling shapes in order to create full rows, the goal of the game being to fill up rows. Epistemic actions are also physical actions performed in the world, but their goal is different: their aim is to gain information by performing computations externally. In Tetris, the epistemic actions consist in rotating the shape in order to decide where and in which orientation to position a falling shape. The main advantages of relying on epistemic actions in Tetris is that rotating the shape in the game is less cognitively costly and more reliable than trying to mentally rotate the shape in the mind. Kirsh and Maglio argue that epistemic actions play an important role in many human activities, and that acting in the environment to gain information, knowledge, and understanding is an essential part of human cognition.

One area of contemporary mathematics were diagrams are omnipresent is knot theory.[7] De Toffoli and Giardino (2014) have conducted a detailed analysis of the use of knot diagrams in the practice of knot theory. They have argued that knot diagrams are entities that support epistemic actions such as computations and inferences. These epistemic actions require a certain cognitive faculty on the part of the agent that they called "manipulative imagination" which, they suggest, is akin to the sort of concrete manipulations one can perform on deformable objects. The moves that can be performed on knot diagrams are codified in the mathematical theory. Accordingly, agents need to be appropriately trained not only to recognize the possible or legitimate moves one can perform on a knot diagram but also to be able to perform these moves. De Toffoli and Giardino have further argued that this manipulative imagination also plays a role in the practice of low-dimensional topology (see De Toffoli and Giardino 2015). Here again, the agent must be able to recognize the permissible actions that can be performed on a given representation, and so need to be appropriately trained to do so. To show this, they have conducted a detailed analysis of Rolfsen's proof of the equivalence of two presentations of the Poincaré homology sphere.

mathematical structures provides a new perspective on mathematical structuralism.

7. Knot theory is a branch of topology that studies knots—a mathematical knot is defined as an embedding of a circle in $\mathbb{R}^3$.

In this particular case, some of the permissible actions consist of continuous transformations. According to De Toffoli and Giardino, this faculty of manipulative imagination builds on pre-existing spatial and motor cognitive capacities but needs to be appropriately trained to play its epistemic function in the practice of knot theory and low-dimensional topology.

Mathematical symbols are artefacts present in virtually every branch of mathematics. De Cruz and De Smedt (2013) have made the case that a primary function of mathematical symbols is to support epistemic actions. These epistemic actions consist in typical symbol manipulations such as those related to negative and imaginary numbers. These symbol manipulations are governed by specific rules associated to the relevant symbolic systems, rules that will need to be learned by the agents. Once acquired, they will provide the agents with an efficient way to perform various forms of mathematical thinking. De Cruz and De Smedt go even further and argue that mathematical symbols support forms of mathematical cognition that would not be possible without resorting to external representations, a view that they articulate by building on the so-called extended mind thesis (Clark and Chalmers 1998; Clark 2008). Their main source of evidence for this claim comes from the history of mathematics. One of their key examples is the case of negative numbers which have received strong oppositions in the past by distinguished mathematicians such as Vieta, Pascal, and De Morgan. De Cruz and De Smedt argue that the invention of the negative numbers, which appears as counter-intuitive, was made possible because of the symbolization associated to the minus sign and the operation of subtraction, a symbolic system with which one can calculate.

This perspective naturally raises the question of what form of cognition underlies thinking and acting with mathematical symbols. Landy, Allen, and Zednik (2014) have proposed a cognitive theory of symbolic reasoning—called the *perceptual manipulations theory*—that attributes central roles to perception and action. According to this theory, external symbols and notations are treated like physical objects by the cognitive system, that is, objects that can be perceived and manipulated. Symbolic reasoning would then rely on a wide range of sensorimotor abilities such as affordance learning, pattern-matching, object tracking, symmetry detection, etc. This theory could explain the importance of careful and effective design of mathematical notations. More specifically, a well-designed notational system is, from this perspective, one that can take advantage of the sensorimotor system capacities in order, for instance, to encourage valid manipulations and refrain invalid one, or to facilitate the application of structurally similar rules from one domain to another.

Giardino (2018) has advanced an encompassing framework aiming to account for different types of mathematical artefacts, including both mathematical diagrams and mathematical symbols. Her proposal is to conceive mathematical artefacts as representational cognitive tools whose main characteristic is to play the double function of representation and instrument. As instrument, the main function of representational cognitive tools is to carry out inferences, i.e., epistemic actions. Giardino offers an account on how epistemic actions operate on representational cognitive tools by building on the notions of *material anchor* from Hutchins (2005) and *affordances* from Gibson (1979). Thinking of representational cognitive tools as material anchors amounts to considering them as material entities which have been designed so that constraints associated to their epistemic functions, as well as to what they represent conceptually, are built in the artefacts themselves. The idea of affordances is that the epistemic actions than can be performed on a given representational cognitive tool are those that are afforded by the tool. These affordances depend on the context or practice in which the cognitive tool is used, which means that agents need to be trained to recognize these affordances, a necessary prerequisite to be able to perform legitimate epistemic actions with the tool.

What these different contributions show is that accounting for what we do with mathematical artifacts in mathematical practice naturally leads to considerations of mental and epistemic actions, and thus of mental and epistemic agency, and requires to pay attention to the cognitive underpinnings of the mental activities under consideration. There is thus room on these issues for fruitful interactions between the philosophy of mathematical practice and related developments in the philosophy of mind, philosophy of action, and cognitive science.

5 Agency and mathematical proofs

Interacting with mathematical proofs is an essential part of what mathematical agents do. This includes evaluating, verifying, communicating, explaining, understanding, and reframing mathematical proofs, among others. But mathematical proofs themselves can also be conceived as being primarily about action. Hamami and Morris (2021) proposed the correspondence displayed in figure 1 between the static notions of deductive step and mathematical proof and the dynamic notions of deductive inference and proof activity which belong to the realm of action. That deductive inferences are first and foremost actions of an epistemic nature has been emphasized by several logicians and philosophers (see, e.g., Sundholm 2012; Prawitz 2012; Boghossian 2014; Wright 2014). Hamami and Morris introduced the term *proof activity* to refer to the sequence of deductive inferences corresponding to a mathematical proof. In this section, we will review contributions that propose an action-based perspective on mathematical proofs, focusing in turn on the notions of deductive inference and proof activity.

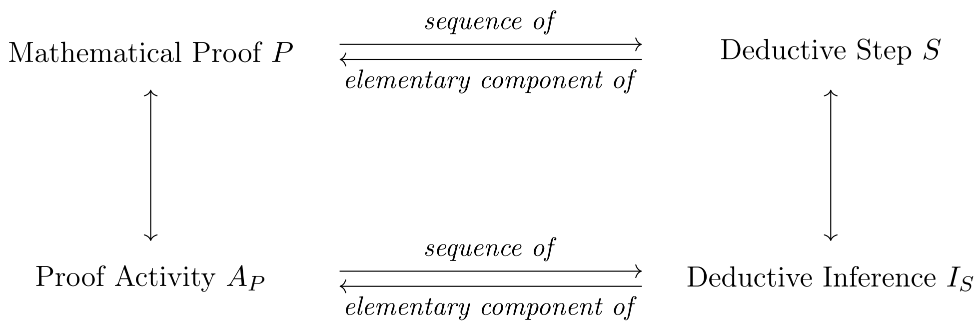

Figure 1: Mathematical proof, proof activity, deductive step, deductive inference

Larvor (2012) has argued that, in order to better understand the nature of informal proofs, it is not sufficient to approach them as bodies of propositions connected in specific ways. Rather, one should see a proof as a sequence of dynamic transitions, that is, one should focus on *inferential actions*. This change of perspective is particularly fruitful because there is no reason to think of inferential actions has being restricted to linguistic representations—inferential actions can operate on different kinds of representations such as diagrams, symbolic expressions, physical models, computer models, etc.[8] Larvor builds on this observation to argue that the validity or invalidity of informal proofs can depend on their content. The main reason being advanced is that inferential actions are not always possible in all domains; only inferential actions that consist of logical inferences are. This means that the validity of these inferential actions must depend somehow on features of the domain under consideration. Larvor argues that inferential actions that are context-dependent can perfectly be rigorous insofar as they come with means of control which govern the inferential actions that are possible and legitimate in a given domain. In this respect, Manders' analysis of Euclid's diagram-based geometric practice (Manders 2008) is an archetypal example of a study aiming to make explicit the means of control governing inferential actions operating on text and diagram in the context of Euclid's geometric proofs. Larvor points out that this perspective shapes and organizes a research program on informal proofs in the philosophy of mathematical practice: the task is to identify the various types of inferential actions, to isolate the features of the domain on which they operate that are responsible for their

8. There is here a direct connection with the topic of agency and mathematical artefacts discussed in the previous section.

validity, and to identify the means of control that govern them.[9]

If it is particularly fruitful to approach individual deductive steps as inferential actions, then it must be equally fruitful to approach entire mathematical proofs as sequences of inferential actions, i.e., as *proof activities* (see figure 1). Hamami and Morris (2021) have undertaken a detailed analysis of proof activities and the form of agency underlying them. Their investigations start from the observation that proof activities have two noticeable characteristics, namely they are goal-directed and temporally extended activities. They are goal-directed in the sense that any proof activity is always directed towards a specific goal, namely to prove or establish the mathematical proposition at hand. They are temporally extended in the sense that any action in a proof activity depends on what happened before and constraints what will happen next, that is, any action is fully integrated into the temporal structure of the activity taken as a whole. In this respect, proof activities are similar to many of our ordinary activities such as travelling or cooking which are obviously goal-directed and which unfold over time. Now, the philosopher of action Michael Bratman (1987) has observed that, for human agents, the realization of goal-directed and temporally extended activities most often requires a form of planning agency. This means that the agents engaged in goal-directed and temporally extended activities are guided by a plan that they construct, revise, and execute over time. In the case of proof activities, this means that the sequence of actions does not come from nowhere but is the result of the execution of a plan that has been constructed rationally. This has some interesting consequences for the epistemology of mathematical proofs in practice. First, it means that mathematical proofs possess what Poincaré (1908) called an *architecture* or a *unity* that can be grasped as a whole (see also Detlefsen 1992). This is due to the fact a mathematical proof is the result of the execution of a plan produced by a mathematical agent, in the same way as a building is the result of executing the plan of an architect. Second, because proof plans are rational constructs, this also means that mathematical proofs produced by rational planning agents possess what Mac Lane (1935) called a *rational structure*—a mathematical proof in practice is not a "mere" sequence of deductive steps. Hamami and Morris (2022a) have argued that this notion of rational structure can be fleshed out by identifying the norms of rationality governing the construction of proof plans. Third, if we accept proof plans as meaningful epistemological entities, then we may expect that some of the activities carried out with proofs will have to do with proof plans. Indeed, several authors have pointed out an intimate connection between proof plans and proof understanding (e.g., Poincaré 1908; Robinson 2000; Folina 2018). Following along this line, Hamami and Morris (2022b) have advanced an account of proof understanding according to which understanding a mathematical proof amounts to being able to rationally reconstruct the proof's underlying plan. Finally, Hamami and Morris (2021) have also argued that proof plans play an important role in the activities of presenting and communicating mathematical proofs.

The action-based perspective on mathematical proofs fits very naturally with the general idea of approaching mathematics as a human activity. As we saw in this section, the static and agentless notions of deductive steps and mathematical proofs can profitably be analyzed through their active counterpart, namely the notions of deductive inferences or inferential actions and proof activities. This perspective opens the way to a logic and epistemology of mathematical proofs particularly suited to account for the spectrum of activities that mathematical agents engaged with in practice in relation to mathematical proofs. As an illustration, we saw here that Larvor's view on inferential actions naturally leads to an account of what it means to evaluate the validity of mathematical proofs, while Hamami and Morris' analysis of proof activities can be used to shape an account of what it means to understand, present, and communicate mathematical proofs. But these are only some of the many activities related to mathematical proofs. An obvious research program to foster this action-based perspective will be to pursue and extend the analysis these different activities. This, in turn, can shed light on the nature of agency involved when interacting with mathematical proofs.

9. This line is further pursued in Larvor (2019).

6 Agency and mathematical texts

Mathematical texts are one of the privileged vehicle for the transmission of mathematical knowledge.[10] If we approach the nature of mathematics through the lens of mathematical practice, a key issue is then to understand how mathematical agents can realize mathematical actions and activities on the basis of mathematical texts. In this section, we will review studies in the field that address exactly this question. As we shall see, such investigations can yield important insights on key questions in the philosophy of mathematical practice.

Computing and proving are two archetypical mathematical activities. The former is central to certain mathematical traditions such as those of Mesopotamia, ancient China, and the Indian subcontinent, while the latter has been privileged in other traditions such as that of ancient Greece (Chemla 2012). Some commentators have downplayed the activity of computing as compared to that of proving for several reasons (e.g., Hacking 2000), a main one being that computations can be carried out blindly, in a step-by-step fashion, without any form of mathematical understanding. Chemla (2015) argues against this view by undertaking a detailed investigation of how texts of mathematical procedures in ancient China prescribe mathematical actions. The focus is on two key mathematical texts, namely the *Writings on mathematical procedures* and the *The Nine Chapters on Mathematical Procedures*. Her analysis yield two important results. The first one is that procedures in these texts cannot be executed in a step-by-step fashion because the specification of certain steps require to attend to other instructions appearing later on in the sequence. This means that a certain form of knowledge is required to "circulate"—Chemla's term—within these texts of procedures. This circulation is necessary to infer from the text the actual sequence of actions to be carried out in specific cases or situations. Chemla argues that this feature witnesses to the generality of the procedures—i.e., the fact that the same procedure can be applied to a wide range of different cases and situations. It is thus essential that the reader be able to infer from the text which actions to take in her specific situation, a competence that goes beyond a blind step-by-step following of the procedure. The second result is that, to infer the sequence of actions to be carried out, the reader needs a certain understanding of the reason(s) why certain steps are carried out. This follows from a meticulous analysis of what it takes to interpret steps in procedures that involve the term 'likewise', that is, steps that require an understanding of what exactly is to be replicated. Taken together, this analysis of the way actions are prescribed in texts of mathematical procedures challenges, according to Chemla, the view that computations are mathematical activities that are carried out blindly without any understanding.

Tanswell (2019) has offered an analysis of imperatives in mathematical proofs—introduced by terms such as "let", "assume", "solve", "observe", etc.[11]—which, he argues, yields important insights into the nature of informal proofs. Imperatives issue instructions or commands to carry out mathematical actions. Tanswell identifies three kinds of imperatives in mathematical proofs. The first kind are imperatives which directly refer to the activity to be carried out, for instance "solve" this equation, "differentiate" or "integrate" this function, "multiply" these two matrices, etc. The second kind are imperatives corresponding to standard instructions, but for which some information is left implicit, such as an "assume" clause without stating exactly what the goal is (e.g., to establish a conditional, to reach a contradiction, to do a reasoning by cases, etc.). The third kind are imperatives whose formulation does not correspond exactly to what is to be done, and for this reason, require background and expertise to be interpreted appropriately. As an example, Tanswell mentions a situation where one is told to "using the Axiom of Choice,

10. This is not to say that mathematical knowledge cannot be transmitted by other means. Agent-to-agent communications through oral transmissions, eventually accompanied by gestures and external representations, has been historically dominant in some mathematical cultures and is certainly very important in present-day mathematical practice. Thanks to José Ferreirós for suggesting this clarification.

11. See the chapter by Inglis and Tanswell in this Handbook (Inglis and Tanswell 2022) for a corpus analysis of imperatives and instructions as they occur in mathematical texts.

for each n choose an enumeration", while "the whole point of the Axiom of Choice is that we cannot actually go about choosing infinitely many times" (Tanswell 2019, p. 7). Acknowledging the importance of imperatives in mathematical proofs motivates, according to Tanswell, what he calls the *recipe model* of informal proofs.[12] The model is introduced through a number of analogies between proofs and ordinary cooking recipes. It is noted that proofs, like cooking recipes, (1) employ the imperative mood, (2) are secondary to the associated activities, (3) involve a clear distinction between the process the author(s) went through to produce them and the way it is being used by the readers or consumers. Tanswell argues that this recipe model of informal proofs is particularly adapted to account for the role of diagrams in proofs for the main reason that diagrams are used to provide instructions in many contexts, and so there is no particular reason to see diagrams as more problematic than texts when it comes to yield instructions.

The studies by Chemla (2015) and Tanswell (2019) show that much is to be learned by conducting detailed analyses of how mathematical texts can lead to mathematical actions and activities. Here, we have seen that this can provide evidence for revising received views on the relation between computation and proof as well as on the nature of informal proof.[13] But, more generally, such investigations are a privileged way to better understand mathematical activities themselves. In fact, for many (most?) mathematical traditions, mathematical texts are all we have. A key task for the historians of mathematics is thus to reconstruct past mathematical activities on the basis of the mathematical texts that came down to us. As Chemla's study illustrates here, this should not only consist of identifying the actions constituting these activities, it should also amount to characterize the knowledge, competence, and understanding that mathematical agents need to possess to carry out the relevant mathematical activities. From this perspective, the issue of mathematical agency necessarily arises whenever one undertakes to analyze or reconstruct mathematical activities, and more generally mathematical practices, on the basis of mathematical texts.

7 Conclusion

What does it mean to *do* mathematics? Doing mathematics obviously involves doing a wide range of different things, and so the only way to progress on this question is to decompose the problem into chunks amenable to philosophical investigations. As we saw in this chapter, philosophers and historians of mathematical practice have made significant progress in this direction by investigating what it means to carrying out mathematical activities, doing things with mathematical artefacts, engaging with mathematical proofs, and performing mathematical actions prescribed by mathematical texts. But this may only be the tip of the iceberg, and it is likely that the question of agency in mathematical practice covers a vast research territory that remains to be uncovered and explored. In this conclusion, I will suggest potential avenues for future work in this direction.

Perhaps one of the most pressing issue is to reflect on what mathematical agents are, a problem that should be addressed for every specific mathematical activity and practice. One of the few authors who have tackled this question directly is Ferreirós (2016), the notion of mathematical agent being central to the philosophical view he advances.[14] For Ferreirós, mathematical agents are first and foremost human agents with ordinary cognitive abilities and limitations, and immersed in a physical, social, and cultural world. Ferreirós argues that all those aspects are

12. See Weber and Tanswell (2022) for further developments of this model and its applications to issues in mathematics education.

13. In this respect, the studies by Chemla and Tanswell fit perfectly within the action-based perspective on mathematical proofs discussed in the previous section.

14. In Ferreirós' words: "The thesis that multiple practices coexist and are interrelated, in a way that is crucial for the conformation of mathematical knowledge, has one interesting consequence. It means that my analysis of mathematical knowledge has to be crucially centered on the agents" (Ferreirós 2016, p. 59).

relevant to his account of mathematical knowledge and mathematical practices. Taken together, the contributions reviewed in this chapter already highlight many salient aspects of mathematical agents. In addition to the features identified by Ferreirós, mathematical agents are also planning agents (Hamami and Morris 2021, 2022a), powered with a faculty of manipulative imagination (De Toffoli and Giardino 2014, 2015, 2016), capable of performing inferential actions (Larvor 2012, 2019), grasping and understanding the reasons behind mathematical actions (Chemla 2015), and following recipes (Tanswell 2019; Weber and Tanswell 2022). A challenge for the philosophy of mathematical practice is to develop an account of mathematical agents capable of articulating these different dimensions in a systematic way.

Research on agency in mathematical practice has so far focused mainly on individual human agency, but several other forms of agency are present in mathematical practice—the most obvious ones are social agency, computer agency, and extended agency. Social agency is present whenever mathematical agents are doing things together, typically when several mathematicians are collaborating to prove a theorem. This raises the question of what it means for a *group* of mathematical agents to do mathematics *together*. As philosophers of action have shown,[15] social agency most often require subtle mechanisms of coordination and interaction. Investigating these mechanisms in the context of mathematical practice will certainly lead important insights into the social dimension of mathematical practice. Additionally, it may be interesting to investigate the agency displayed by the mathematical community or subgroups thereof, for instance when attributing credits and rewards (see, e.g., Jaffe and Quinn 1993; Rittberg, Tanswell, and Van Bendegem 2020). Similarly, the increasing role of computers in mathematical practice raises the questions of what it means *for* computers to do mathematics, if at all, and what it means to do mathematics *with* computers.[16] These issues pop up concretely whenever one investigates cases of proving or discovering where computers are involved in an essential way—e.g., in the computer-assisted proof of the four color theorem. Finally, the many studies on mathematical artefacts in mathematical practice raises the question of what exactly the agent who does mathematics consists in. When a mathematical activity is carried out by a human agent relying on some mathematical artefacts, what is doing mathematics is a sort of integrated system composed of the coupling of the agent and the artefacts, in which case the mathematical agent may better be conceived as an extended agent. Such an approach could be developed, for instance, in the line of the cyborg conception of agency proposed by Clark (2003), following the famous extended mind thesis much discussed in the philosophy of mind (Clark and Chalmers 1998). Another challenge for philosophers and historians of mathematical practice is thus to identify and characterize the different forms of agency at play in mathematical practice.

The question of agency in mathematical practice is both a theoretical and a methodological issue for the philosophy of mathematical practice. It is a theoretical issue in the sense that any account of what it means to do mathematics may be expected to say something about the agents doing mathematics. But it is also a methodological issue in the sense that our underlying conception of mathematical agency guides, in part, our investigations into specific mathematical activities and practices. More specifically, it provides us with a "template" to address questions such as: What does this mathematical activity or practice consists in? What does it take for an agent to be able to properly engage in it? These two aspects are intimately intertwined: progress on the theoretical front may yield new tools and perspectives to investigate specific mathematical activities and practices;[17] in turn, these investigations may yield new empirical data to constrain and revise our theoretical conceptions. Given that the nature of agency is one of the outstanding issues spanning the Humanities, progress in the case of mathematical agency can certainly be made by recruiting concepts and resources from other fields, both within

15. See Roth (2017) for a review of the literature on social agency in the philosophy of action.

16. See Avigad (2008) for some investigations in this direction.

17. A perfect illustration of this is provided by the studies in Chemla and Virbel (2015) which build, in part, on the theoretical framework developed by Virbel and colleagues (Grandaty, Debanc, and Virbel 2000; Virbel 2000).

philosophy—especially from the philosophy of action, philosophy of mind, epistemology, and the philosophy of science—but also from other areas of the social sciences such as sociology and anthropology. Understanding the nature of agency in mathematical practice is thus likely to require the full interdisciplinary perspective that was claimed as a characteristic feature of the philosophy of mathematical practice.

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