Rationality in Mathematical Proofs

Final Version Submitted to the *Australasian Journal of Philosophy*

Abstract

Mathematical proofs are not sequences of *arbitrary* deductive steps—each deductive step is, to some extent, *rational*. This paper aims to identify and characterize the particular form of rationality at play in mathematical proofs. The approach adopted consists in viewing mathematical proofs as reports of *proof activities*—i.e., sequences of deductive inferences—and in characterizing the rationality of the former in terms of that of the latter. It is argued that proof activities are governed by specific norms of rational planning agency, and that a deductive step in a mathematical proof qualifies as rational whenever the corresponding deductive inference in the associated proof activity figures in a plan that has been constructed rationally. It is then shown that mathematical proofs whose associated proof activities violate these norms are likely to be judged as defective by mathematical agents, thereby providing evidence that these norms are indeed present in mathematical practice. We conclude that, if mathematical proofs are not mere sequences of deductive steps, if they possess a rational structure, it is because they are the product of rational planning agents.

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1 Introduction

On the traditional view, a *mathematical proof*[1] is a sequence of deductive steps, the only requirement being that each deductive step be valid. Yet, several leading mathematicians, who also entertain philosophical interest in the methodological issues of their discipline, have pointed out that there is *more* to mathematical proofs. Henri Poincaré and Saunders Mac Lane are cases in point when they wrote:

> A mathematical demonstration is not a simple juxtaposition of syllogisms; it consists of syllogisms *placed in a certain order*, and the order in which these elements are placed is much more important than the elements themselves. [Poincaré 1908, p. 49]

> A proof for a given theorem is not a haphazard collection of individual steps, taken arbitrarily one after another, as the classical logic might easily lead us to believe. On the contrary, there is some definite reason for the inclusion of each one of these steps in the proof; that is, each individual step is taken for some specific purpose. [Mac Lane 1935, p. 125]

So what exactly is missing from the traditional picture? Answering this question would help advance our understanding of the nature of mathematical proofs in mathematical practice.[2] The aim of this paper is to address this issue by identifying what this missing element might be and by proposing a possible characterization of it.[3]

[1]Throughout this paper, we will use the term 'mathematical proof' to refer to proofs in ordinary mathematical practice.

[2]This is currently one of the main issues on the research agenda of the philosophy of mathematics. For an overview of this literature, see Hamami and Morris [2020, section 4.1] and Macbeth [2021] as well as the references cited therein.

[3]Of course, this is not to suggest that providing such a characterization would *exhaust* the

One way of making sense of Poincaré's and Mac Lane's remarks is to recognize that proofs in mathematical practice are the products of *rational agents* pursuing specific *epistemic goals*. From this perspective, a written mathematical proof appears as a *report* of a successful sequence of epistemic actions—namely deductive inferences—which have been taken towards an epistemic goal—namely establishing a given mathematical proposition—potentially under certain constraints. In this respect, a mathematical proof is analogous to a travel diary in which an agent has recorded the successful sequence of moves that brought her from her starting point to her final destination. Now, if we characterize a travel diary as a "simple juxtaposition" or an "haphazard collection" of moves, then surely we miss something essential. The reason is that a journey is not a sequence of *arbitrary* moves, otherwise we would very rarely reach our final destination. Rather, we would expect that each move was the result of a *rational* decision by the agent who undertook them. The exact same observation holds for mathematical proofs. A mathematical proof is not the report of a sequence of *arbitrary* deductive inferences, otherwise we would very rarely succeed in proving the mathematical propositions we intend to establish. Rather, we would expect that a mathematical proof reports a sequence of *rational* epistemic actions directed towards the goal of establishing the mathematical proposition at hand. This may explain why Poincaré and Mac Lane reject the idea that a mathematical proof is a "simple juxtaposition" or an "haphazard collection" of deductive steps.

The perspective just sketched suggests one possible way to get at what is missing in the traditional view, namely to characterize what it means for a deductive step in a mathematical proof to qualify as *rational*.[4] If we consider that a mathematical proof is a report of what we will call a *proof activity*—i.e., a sequence of deductive

question of the nature of mathematical proofs in mathematical practice.

[4]This is in direct line with Mac Lane [1935, section 5] who suggests that there is always a "reason" behind each deductive step of a mathematical proof.

inferences—then the task boils down to characterizing what it means for a deductive inference in a proof activity to qualify as rational. In other words, the task is to identify the *norms of rationality* that govern a certain type of action in a certain type of activity. In previous work [Hamami and Morris 2021], we have provided an analysis of proof activities and have argued that their realization by human agents with limited cognitive capacities requires a form of planning agency as theorized by Michael Bratman [1987]. Building on this analysis, we will propose in this paper that deductive inferences in proof activities are governed by norms of *planning rationality*. We will then argue that this yields a plausible analysis of what Poincaré and Mac Lane consider to be missing from the traditional view of mathematical proof.

This paper is structured as follows. In section 2, we introduce the notion of *proof activity* and argue that the rationality of deductive steps in mathematical proofs can be characterized in terms of the rationality of deductive inferences in proof activities. In section 3 we argue that proof activities require a kind of planning agency and spell out the core components of our account of such agency. Then, in section 4, we identify the norms of planning rationality governing proof activities and argue that these norms allow us to characterize the rationality of deductive steps in mathematical proofs. Sections 5 and 6 provide evidence that these norms are at play in mathematical practice by showing that proofs whose associated proof activities violate them are judged as defective by practitioners. Section 7 concludes the paper with a discussion of the *rational structure* of mathematical proofs, a notion introduced by Mac Lane [1935, p. 125] which, we suggest, can be generalized to other kinds of mathematical developments.

2 Mathematical Proofs and Proof Activities

The term *mathematical proof* usually refers to the written mathematical proofs commonly found in mathematical textbooks, articles, and monographs. As such, a mathematical proof is a static, agent-free object which consists of a sequence of *deductive steps.* But as noted by several philosophers and logicians [see, e.g., Sundholm 2012; Boghossian 2014], the notion of deductive step has a direct counterpart in the realm of action with the notion of *deductive inference.* A deductive inference is an epistemic action that can bring an agent from one epistemic state to another, for instance, from a state of knowing or believing the premisses to one of knowing or believing the conclusion.

Following this line of thought, we can introduce the notion of *proof activity* as the direct counterpart of the notion of mathematical proof in the realm of action. Thus, in the same way as a mathematical proof consists of a sequence of deductive steps, a proof activity consists of a sequence of deductive inferences. Furthermore, to any mathematical proof is associated a given proof activity in which each deductive inference corresponds to a deductive step in the original mathematical proof, in such a way that the overall structure of the sequence is preserved.[5] This means that there is always an isomorphic correspondence between a mathematical proof and its corresponding proof activity.[6]

As mentioned previously, a (written) mathematical proof appears, from this perspective, as a report of its associated proof activity. In this respect, it is analogous to

[5]A proof activity is thus different from the activity of *searching* for a proof. The latter will involve various trials and errors, dead ends, explorations, changes of proof strategy, etc. In this work, we are concerned with analyzing the rationality of mathematical proofs as a way of getting at what Poincaré and Mac Lane considers to be missing from the traditional picture. We are not concerned with analyzing the rationality of the proof discovery process, although the two issues are intimately connected.

[6]Proof activities are what we take to be the object of rationality evaluation when we evaluate the rationality of mathematical proofs. It is a technical notion that does not necessarily match the sequence of inferences carried out by the author of the proof—because she may not have reported all the inferences she has carried out, for instance when leaving details to the reader—nor the sequence of inferences carried out by the reader—because she may have carried out more inferences than the one listed in the written mathematical proof, for instance when filling in the details left to the reader.

other reports of past activities such as, for instance, a scientific report of a completed experiment, or a police or newspaper report of some people's doings. Now, we are often led to judge the actions recorded in such reports, for instance, when a court evaluates whether an action was legal or illegal from a police report, or a reader evaluates whether a politician's actions were rational or irrational from a newspaper article. In all these cases, we are judging a particular action, in a particular activity, as carried out by a particular agent. Our proposal is that this is exactly what happens when we judge the rationality of a deductive step in a mathematical proof: we are judging the rationality of a deductive inference, in a proof activity, as carried out by a mathematical agent. In analyzing the rationality of mathematical proofs along these lines, we are then following John Broome [2010, 2013] in considering the rationality of mathematical proofs to be derivative from the rationality of mathematical agents:

> The word 'rationality' often refers to a property—the property of being rational. This property may be possessed by people, and also by beliefs, acts, conversations, traffic schemes and other things. I shall concentrate on the rationality of people. The rationality of other things is derivative from the rationality of people. [Broome 2010, p. 285]

These considerations can be summed up in the following characterization:

A deductive step S at stage s in a mathematical proof P is *rational*

$$\Leftrightarrow$$

It is rational for a mathematical agent to carry out the deductive inference I_S at stage s in the proof activity A_P,

where I_S and A_P are the deductive inference and the proof activity corresponding to the deductive step S and the mathematical proof P. If we are to specify this characterization further, that is, to specify what it means for a deductive inference in a proof activity to qualify as rational, we then need a precise conception of the kind of agency involved in proof activities. This is the issue that we now turn to.

3 Planning Agency in Proof Activities

Proof activities are always directed towards a specific goal, namely to establish the mathematical proposition at hand. They are also temporally extended in the sense that each deductive inference stands in a particular relation to the inferences that are prior and posterior to it in the activity, that is, each deductive inference is inscribed within the overall temporal structure of the activity. Now, as noted by Michael Bratman [1987], the realization of goal-directed and temporally extended activities for cognitively limited beings like us most often requires a form of planning agency. In Hamami and Morris [2021], we have provided an account of planning agency in proof activities based on Bratman's theory of planning agency.[7] This account was obtained by specifying the notions of intention, practical reasoning, and plan in the context of proof activities. In this section, we recall the main elements of our account.

3.1 Intentions in Proof Activities

In Bratman's theory of planning agency, *intentions* are the building blocks of plans. For instance, your plan to spend a week in New York during the summer is or will be composed of several intentions regarding how you will get there, where you will stay, what you will visit, etc. Similarly, your plan to prove a given mathematical proposition is or will be composed of several intentions to prove a number of other mathematical propositions. As it turns out, intentions in the context of proof activ-

[7]In this previous work, our aim was to provide a precise notion of plan for mathematical proofs. It is very common in mathematical practice to refer to the plan of a proof. This notion of plan was also identified by Mac Lane [1934, 1935] as a key element in analyzing the structure of mathematical proofs.

ities always take the same form, namely one always intends *to show*—equivalently, to prove, to establish—a given mathematical proposition *from* other mathematical propositions. In our account, we refer to them as *proving intentions* which we denote using the following sequent notation:

$$P_1, \ldots, P_n \;\Rightarrow\; C,$$

where $P_1, \ldots, P_n$ and C are placeholders for ordinary mathematical propositions—$P_1, \ldots, P_n$ and C are referred to as the *hypotheses* and the *conclusion* of the considered proving intention. For example, if a mathematics student intends to prove the Intermediate Value Theorem as an exercise, her proving intention will be written as:

$$f : [a, b] \to \mathbb{R} \text{ is a continuous function } \;\Rightarrow f \text{ takes on each value between } f(a) \text{ and } f(b),$$

and if a research mathematician intends to show that Goldbach's conjecture is true, her proving intention will be written as:

$$k \text{ is an even number greater than or equal to } 4 \;\Rightarrow\; k = p + q \text{ where } p, q \text{ are prime.}$$

We then introduced a further distinction between proving intentions of type *'to show'* and proving intentions of type *'to infer'*. The difference is the following: a proving intention of type 'to show' cannot be fulfilled directly and so will always have to be turned into further proving intentions; a proving intention of type 'to infer' is an intention to carry out a deductive inference and so can be fulfilled directly by actually carrying out the considered inference, i.e., by performing an action. A proving intention $P_1, \ldots, P_n \;\Rightarrow\; C$ of type 'to show' should be read as the intention

to show C *from* $[P_1, \ldots, P_n]$,

while a proving intention of type 'to infer' should be read as the intention

to infer C *from* $[P_1, \ldots, P_n]$.

Proving intentions of types ‘to show’ and ‘to infer’ constitute the elementary components of plans for proof activities.

3.2 Practical Reasoning in Proof Activities

Plans are built over time through *practical reasoning*. If you intend to spend a week in New York during the summer, you will then be led to reason from this initial intention to more specific intentions regarding the mode of transport to adopt, the company to travel with, the exact time to depart, etc. Similarly, at the very beginning of a proof activity, one always starts with the proving intention to establish the mathematical proposition at hand,[8] which is then turned into more specific proving intentions as the proof activity proceeds. In the context of proof activities, practical reasoning consists then in reasoning from a proving intention to one or more proving intentions arranged in a subplan. This process is best illustrated by some examples. Suppose that you intend to show that a given relation $\sim$ is an equivalence relation on a set S from some propositions $P_1, \ldots, P_n$. One possible way to proceed is to reason from this intention to the following subplan:

To show that $\sim$ is an equivalence relation on S, *from* $[P_1, \ldots, P_n]$:

1. **Show** $\sim$ is reflexive, *from* $[P_1, \ldots, P_n]$,
2. **Show** $\sim$ is symmetric, *from* $[P_1, \ldots, P_n]$,
3. **Show** $\sim$ is transitive, *from* $[P_1, \ldots, P_n]$,
4. **Infer** $\sim$ is an equivalence relation, *from* $\sim$ is reflexive *and* $\sim$ is symmetric *and* $\sim$ is transitive.

In most cases, there will be several different ways in which you could transform a

[8]By contrast, in the activity of *searching* for a proof, this initial proving intention may change as the proof search proceeds since one may be led to reformulate the theorem one intends to prove, as famously shown by Lakatos [1976].

proving intention in the course of a proof activity. For instance, if you intend to show that $\forall n H(n)$ where n ranges over the natural numbers, then you can proceed by mathematical induction, in which case you will produce the following instance of practical reasoning:

To show $\forall n H(n)$, *from* $[P_1, \ldots, P_n]$:

1. **Show** $H(0)$, *from* $[P_1, \ldots, P_n]$,
2. **Show** $\forall p(H(p) \to H(p+1))$, *from* $[P_1, \ldots, P_n]$,
3. **Infer** $\forall n H(n)$, *from* $H(0)$ *and* $\forall p(H(p) \to H(p+1))$.

But you can also proceed by showing that $H(n)$ holds for both the odd and the even numbers:

To show $\forall n H(n)$, *from* $[P_1, \ldots, P_n]$:

1. **Show** $\forall n H(2n)$, *from* $[P_1, \ldots, P_n]$,
2. **Show** $\forall n H(2n+1)$, *from* $[P_1, \ldots, P_n]$,
3. **Infer** $\forall n H(n)$, *from* $\forall n H(2n)$ *and* $\forall n H(2n+1)$.

Of course, practical reasoning in proof activities can be much more complicated, involving for instance long chains of trials and errors before a successful subplan can be reached.[9]

3.3 Plans in Proof Activities

In Hamami and Morris [2021], we defined the notion of plan for proof activities as follows: an agent's *plan* for a proof activity is an *ordered tree*[10] such that (1) each *node* is a proving intention, (2) the *root* is the proving intention corresponding to the theorem at hand, and (3) each *set of ordered children* of a given parent node

[9] See Hamami and Morris [2021] for a detailed analysis of several examples of practical reasoning in proof activities at different levels of complexity.

[10] In the mathematical sense of the term: an *ordered tree* is a rooted tree where each node comes equipped with an ordering of its children.

is a subplan that has been obtained from the parent node through an instance of practical reasoning. When an agent *executes* her plan for a proof activity, she will then be led to the actual *realization* of the proof activity, i.e., to the actual *carrying out* of a sequence of deductive inferences. This is directly analogous to what happens when an agent executes her traveling plan: she is led to the actual realization of the traveling activity, i.e., to the actual performance of a sequence of moves. As we shall now see, proof activities are subject to specific norms of rationality relative to the planning agency that gives rise to them.

4 Norms of Rationality for Proof Activities

Recall that our strategy to get at what is missing from the traditional picture of mathematical proof is to characterize what it means for a deductive step in a mathematical proof to qualify as rational. As we discussed earlier, this boils down to characterizing what it means for a deductive inference in a proof activity to qualify as rational, that is, to identifying the *norms of rationality* governing deductive inferences in proof activities. Now, deductive inferences in proof activities are entirely determined by the planning agency that gives rise to them since a proof activity is the result of the execution of its underlying plan. Whether a deductive inference in a proof activity qualifies as rational depends, then, on the potential rationality of the underlying planning agency. This means that the norms of rationality governing deductive inferences in proof activities are directly inherited from the norms of rationality governing the underlying planning agency. In this section, we will identify the latter and use them to formulate the former.

4.1 Norms of Rationality for Planning Agency

Bratman [1987, section 3.2] identifies two norms of rationality for plans that he coins consistency and means-end coherence:

Consistency: Rationality requires of the agent that her plans be strongly consistent, that is, internally consistent and consistent with her beliefs: assuming that all her beliefs are true, it must be possible for her plans to be successfully executed.

Means-End Coherence: Rationality requires of the agent that her plans be filled in with subplans whenever she considers it necessary, at the present time, to ensure that her plans can be successfully executed.

Because plans are often specified further through practical reasoning, to the two norms of rationality for plans correspond two norms of rationality for practical reasoning that we will refer to as consistency* and means-end coherence*. These norms exert rational pressure on practical reasoning to maintain the consistency and means-end coherence of the agent's plans over time.

The norm of consistency* stipulates that, whenever an agent specifies her plans through practical reasoning, she should do so in such a way that the resulting plans still satisfy the norm of consistency:

Consistency*: Rationality requires of the agent that a subplan integrated into her plans as a solution to a problem posed by one or more of her prior intentions preserves the strong consistency of her plans.

To return to the New York trip example, this means that when you decide which means of transport to adopt to go to the airport, you should aim for a solution that results in a plan that, to your beliefs, can be successfully executed. It would be irrational, for instance, to intend to take a train that you believe would not stop at the airport, or would not leave you enough time to reach the gate before it closes.

The norm of means-end coherence* stipulates that the agent shall engage in practical reasoning to specify her plans further whenever this appears to be necessary for them to be successfully executed. This means that, at any given point in time, the agent shall address what we will call her *critical intention(s)* at this time, that is, the intention(s) in her plans that needs to be turned into more specific intentions at this time for her plans to be successfully executed:

Means-End Coherence*: Rationality requires of the agent that she addresses the problem(s) posed by her critical intention(s) at the present time.

In the New York trip example, this means that once you have decided to go by plane and know the date and time of your flight, you must decide how to get to the airport before it's too late to check in and go through security. It would be irrational, for instance, *not* to engage in practical reasoning to decide how to get to the airport if it turns out to be necessary to do so at the present time to catch the flight you intend to take.

4.2 Norms of Rationality for Planning Agency in Proof Activities

In the particular case of proof activities, the agent's plan is *entirely generated* through practical reasoning. This is due to the fact that, at the beginning of any proof activity, the agent's plan is always composed of a *single* intention—the intention of proving the theorem at hand—and so the only possible way for the agent to develop her plan is to engage in practical reasoning. As a consequence, whether the agent's plan satisfies the norms of rationality for plans depends *exclusively* on whether the agent has developed her plans in accordance with the norms of rationality for practical reasoning. In other words, at any given stage of a proof activity, the agent's plan satisfies the norms of consistency and means-end coherence if and only if the agent has developed her plan over time in accordance with the norms of consistency* and means-end coherence*. Thus, to specify the norms of rationality for planning agency in proof activities, it suffices to specify the norms of rationality governing the associated practical reasoning.

The above norms of rationality for practical reasoning can be straightforwardly adapted to the case of proof activities. The norm of $\mathsf{consistency}^*$ says that, whenever an agent integrates a subplan into her plans as a solution to a problem posed by one or more of her prior intentions, she shall do so in such a way as to preserve the strong consistency of her plans, i.e., in such a way that, to her beliefs, her updated plans can be successfully executed. This leads to the following specification of the norm of $\mathsf{consistency}^*$ in the case of proof activities:

$\mathsf{Consistency}^*_{\mathsf{PA}}$: Rationality requires of the agent that the integration into her plan of a subplan obtained through practical reasoning from a problem posed by one of her proving intentions results in a plan that, to her beliefs, can be successfully executed.

The norm of $\mathsf{means\text{-}end\ coherence}^*$ says that the agent shall address the problem posed by her critical intention(s) at the current time.[11] In our account of the planning agency underlying proof activities, the agent always entertains one single critical proving intention at any given stage of a proof activity. The specification of $\mathsf{means\text{-}end\ coherence}^*$ in the case of proof activities can then be formulated in a straightforward way as follows:

$\mathsf{Means\text{-}End\ Coherence}^*_{\mathsf{PA}}$: Rationality requires of the agent that she addresses the problem posed by the proving intention that is critical for her at the current stage.

4.3 Norms of Rationality for Deductive Inferences in Proof Activities

So far, we have identified the norms of rationality for the planning agency underlying proof activities. Since deductive inferences in proof activities are *entirely* determined

[11] Because proof activities are *sequential* processes, we shall talk of the 'stages' of a proof activity, and talk of the current or present 'stage' instead of the current or present 'time'.

by the planning agency that gives rise to them, whether a deductive inference in a proof activity qualifies as rational depends then *exclusively* on the eventual rationality of the underlying planning agency. This relation can be concretely expressed by saying that a deductive inference at a given stage in a proof activity is *rational* whenever it figures in a plan that is itself *rational*, that is, a plan that has been constructed *rationally*. This leads to the following formulation of the norms of rationality governing deductive inferences in proof activities:

$\textsf{Rationality}_{\textsf{DI}\propto\textsf{PA}}$: Rationality requires of the agent that her deductive inference I_S at stage s of her proof activity A_P figures in a plan that she constructed in accordance with the norms of rationality for practical reasoning in proof activities, that is, the norms of $\textsf{consistency}^*_{\textsf{PA}}$ and $\textsf{means-end coherence}^*_{\textsf{PA}}$.

We can now complete our characterization of what it means for a deductive step in a mathematical proof to qualify as *rational*:

A deductive step S at stage s in a mathematical proof P is *rational*

$$\Leftrightarrow$$

A typical mathematical agent satisfies the norm $\textsf{rationality}_{\textsf{DI}\propto\textsf{PA}}$ when carrying out the deductive inference I_S at stage s in the proof activity A_P.

This, we claim, constitutes one possible way to characterize what might be missing from the traditional view of mathematical proofs. Why? Recall that Poincaré and Mac Lane rejected the idea that a mathematical proof is a "simple juxtaposition" or an "haphazard collection" of deductive steps. In other words, they contested the idea that deductive steps in mathematical proofs are *arbitrary*. In the approach developed in this paper, deductive steps in mathematical proofs are not arbitrary because they are always the result of *rational* decisions. These rational decisions are to be found in the instances of practical reasoning that led to the construction of the plan underlying the considered mathematical proof. This is exactly the idea embedded into the above characterization and the norm $\mathsf{rationality}_{\mathsf{DI} \propto \mathsf{PA}}$: a deductive step in a mathematical proof is rational whenever the corresponding deductive inference in the associated proof activity figures in a plan that has been rationally constructed, i.e., constructed using our faculty of practical reason. If this analysis is correct, we shall then be able to find evidence that the norms of rationality identified in this section are indeed at play in mathematical practice. The aim of the next two sections is to assess our proposed characterization along this line.

5 Assessing the Account: Rationality Judgments and Proof Defects

How should we assess whether the norms of rationality just identified are indeed at play in mathematical practice? Bratman has repeatedly emphasized that the norms of consistency and means-end coherence, as well as their counterparts for practical reasoning, are not just "rules of thumb" [Bratman 2017, p. 26]. Instead, an agent violating them will open herself up to charges of "critizable irrationality" [Bratman 1987, p. 37], witnessing a "rational breakdown" [Bratman forthcoming, p. 1]. As a consequence, if the norm $\mathsf{rationality}_{\mathsf{DI} \propto \mathsf{PA}}$ indeed applies to deductive steps in mathematical proofs, then a mathematical proof whose associated proof activity *violates* either the norms of $\mathsf{consistency}^*_{\mathsf{PA}}$ or $\mathsf{means\text{-}end\ coherence}^*_{\mathsf{PA}}$ should be judged as *defective*. We will now argue that this is the case in mathematical practice.

The typical situation in which such rationality judgments occur is when the rationality of a mathematical proof produced by an agent with expertise $\mathcal{E}_a$ (the author)[12] is judged by another agent with expertise $\mathcal{E}_r$ (the reader). In mathematical practice, mathematical agents can differ in terms of their *level(s)* and/or *area(s)* of expertise. For instance, a student and a mathematics professor would differ at least

[12]In mathematical practice, proofs are often produced by groups of agents. To ease the discussion, we will only consider here the single agent case.

in terms of their level of expertise, while an analyst and a topologist would differ at least in terms of their area of expertise. It is then a truism that the writing and the processing of mathematical proofs is highly dependent on the expertise of the agents involved [see, e.g., Thurston 1994, p. 175].

Since we are looking for potential violations of norms of rationality, we are interested in cases in which an agent with expertise $\mathcal{E}_r$ would judge a deductive step in a mathematical proof produced by an agent with expertise $\mathcal{E}_a$ as *not* rational. Although mathematicians would probably not put things in these philosophical terms, such judgments of rationality are relatively common in mathematical practice. To see this, it suffices to reflect on how those judgments usually manifest themselves. In ordinary life, judging an action as not rational would often lead to a form of puzzlement, triggering a remark of the form "I do not see *why* you are doing this". This might happen, for instance, if you see your friend searching for his car keys in the fridge. You may then ask him why he is doing this, to which he may reply that this is where his girlfriend hid the keys to prevent him from driving home last night after the party. Such puzzlement is fairly common when it comes to mathematical proofs, especially when the expertise of the reader is significantly different from that of the author. Thus, a student might not see why her mathematics professor is taking this particular step at this stage of the proof, in the same way as an analyst might not see why her topologist colleague is doing this or that in the course of her proof. In such cases, the reader will be dissatisfied with the proof which may then *appear* as defective to her. Importantly, this does not mean that the reader would take the author of the proof to be irrational. A student would certainly not judge his mathematics professor to be irrational, in the same way as an analyst would certainly not judge his topologist colleague to be irrational. It is only from the perspective of the reader that (some deductive steps in) the proof may not appear as rational. To refine our initial issue, the question is then whether a mathematical proof whose associated proof activity violates either the norms of $\mathsf{consistency}^*_{\mathsf{PA}}$ or $\mathsf{means\text{-}end\ coherence}^*_{\mathsf{PA}}$

from the perspective of a reader with expertise $\mathcal{E}_r$ should appear as defective to *this* particular reader.

What would a violation of the norm of $\mathsf{means\text{-}end\ coherence}^*_{\mathsf{PA}}$ amount to? In Bratman's theory of planning agency, the norm of $\mathsf{means\text{-}end\ coherence}$ is essentially concerned with temporal deadlines, in the sense that, to be rational, an agent ought to fill in her plan *by a given time* if she believes this to be necessary for what she plans to do. A typical example of a violation of $\mathsf{means\text{-}end\ coherence}$ would be a case in which you plan to go to the Rolling Stones concert next summer, you believe that buying a ticket on the day at which they are put on sale is a necessary means to this end, but you do not come up with any specific subplans to buy the tickets before the next day (e.g., by going to a ticket shop or by visiting one online). In the context of proof activities, a violation of $\mathsf{means\text{-}end\ coherence}^*_{\mathsf{PA}}$ would simply mean that you never address what we have called the critical proving intention at a given stage of a proof activity. In this case, your plan will remain incomplete and so the execution of the plan will never lead to a successful proof activity, that is, the corresponding mathematical proof will itself remain necessarily incomplete. Thus, insofar as an incomplete proof is necessarily defective to the extent that it does not establish the mathematical proposition at hand, it is relatively obvious that the norm of $\mathsf{means\text{-}end\ coherence}^*_{\mathsf{PA}}$ is indeed at play in mathematical practice.

Violations of the norm of $\mathsf{consistency}^*_{\mathsf{PA}}$ are more interesting. The key observation is that the norm of $\mathsf{consistency}^*_{\mathsf{PA}}$ depends fundamentally on the *beliefs* of the agent: to be rational, any subplan to be integrated into the agent's overall plan must result in a plan that, *to the agent's beliefs*, can be successfully executed. One consequence of the dependence of $\mathsf{consistency}^*_{\mathsf{PA}}$ on the agent's beliefs is that agents with different expertise can be led to diverging rationality judgments on the same mathematical proof. As we noted above, this is exactly what is to be expected from the point of view of mathematical practice. A typical example of this is when the author is using a method or technique which is not known to the reader. For instance, a topologist might adopt a given subplan in her proof which is just the application of a well-known method commonly used in topology but rarely, if at all, used in analysis. Because of her experience with the method, the topologist might believe that the method, and so the resulting plan, has a chance to succeed, but this might not be the case for the analyst reading the proof. In this situation, all the deductive steps in this subplan will appear as rational to the topologist but not to the analyst.

What would a violation of the norm of $\mathsf{consistency}^*_{\mathsf{PA}}$ amount to in practice? From the perspective of the reader of a proof, a violation of $\mathsf{consistency}^*_{\mathsf{PA}}$ occurs whenever the reader has no reason to believe that a given subplan in the overall plan of the corresponding proof activity can be successfully executed. In our account, the agent should then judge all the deductive inferences figuring in this subplan as defective. That this is indeed the case is better seen by examining some concrete examples. In the next section, we will analyze in detail an example originally proposed and discussed by Pólya [1949].

6 Assessing the Account: A Concrete Example

In this section, we analyze a proof of Carleman's inequality due to Pólya [1925]. Although the proof is perfectly correct, Pólya [1949] points out that many readers will find it unsatisfactory.[13] Here we argue that, if the proof appears defective to many readers, it is because those readers would judge the corresponding proof activity to violate the norm of $\mathsf{consistency}^*_{\mathsf{PA}}$.

Carleman's inequality and its proof are given below exactly as they appear in Pólya [1949, pp. 684–685]:

Theorem. *If the terms of the sequence a_1, a_2, a_3, $\cdots$ are nonnegative*

[13] Pólya's proof has been analyzed from the perspective of mathematical explanation by Sandborg [1997], and from the perspective of motivation by Morris [forthcoming].

real numbers, not all equal to 0*, then*

$$\sum_{1}^{\infty}(a_1 a_2 a_3 \cdots a_n)^{1/n} \; < \; e\sum_{1}^{\infty} a_n.$$

Proof. Define the numbers c_1, c_2, c_3, $\cdots$ by

$$c_1 c_2 c_3 \cdots c_n \; = \; (n+1)^n$$

for $n = 1, 2, 3, \cdots$. We use this definition, then the inequality between the arithmetic and the geometric means, and finally the fact that the sequence defining e, the general term of which is $[(k+1)/k]^k$, is increasing. We obtain

$$\begin{aligned}
\sum_{1}^{\infty}(a_1 a_2 \cdots a_n)^{1/n} &= \sum_{1}^{\infty}\frac{(a_1 c_1 a_2 c_2 \cdots a_n c_n)^{1/n}}{n+1} \\
&\leq \sum_{1}^{\infty}\frac{a_1 c_1 + a_2 c_2 + \cdots + a_n c_n}{n(n+1)} \\
&= \sum_{k=1}^{\infty} a_k c_k \sum_{n\geq k}\frac{1}{n(n+1)} \\
&= \sum_{k=1}^{\infty} a_k c_k \sum_{n=k}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) \\
&= \sum_{k=1}^{\infty} a_k \frac{(k+1)^k}{k^{k-1}}\frac{1}{k} \\
&< e\sum_{k=1}^{\infty} a_k.
\end{aligned}$$

□

According to Pólya, the problem with the above perfectly correct proof lies at the very beginning: the introduction of the c_i sequence. Pólya characterized this as a 'deus ex machina' step and imagined a variety of objections to it:

> "It appears as a rabbit pulled out of a hat."
>
> "It pops up from nowhere. It looks so arbitrary. It has no visible motive

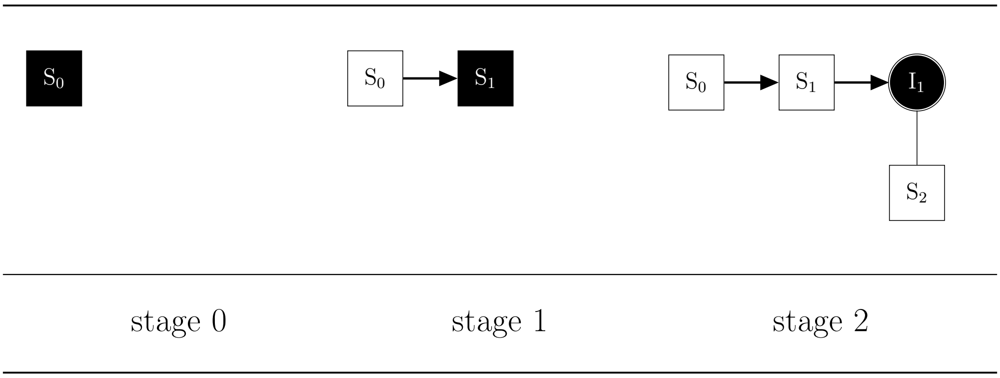

Figure 6.1: Stage by stage progression for the first three stages of the planning underlying the proof activity corresponding to Pólya's proof

> or purpose."
> "I hate to walk in the dark. I hate to take a step, when I cannot see any reason why it should bring me nearer to the goal."
> "Perhaps the author knows the purpose of this step, but I do not and, therefore, I cannot follow him with confidence."
> "This step is not trivial. It seems crucial. If I could see that it has some chances of success, or see some plausible provisional justification for it, then I could also imagine how it was invented and, at any rate, I could follow the subsequent reasoning with more confidence and more understanding." [Pólya 1949, p. 685]

To better understand the objections to the c_i sequence, we analyze, below, the first three stages of planning underlying the proof activity corresponding to Pólya's proof, a graphical representation of which is provided in figure 6.1. In what follows, the planning and proof activity is evaluated from the perspective of a mathematical agent reading Pólya's proof and reconstructing for herself the underlying planning.

At the start of the proof activity corresponding to Pólya's proof, the agent's plan is composed of a single proving intention, which is the intention to show the theorem at hand:

$$\mathrm{S}_0 \; : \begin{array}{l} a_1,\ a_2,\ a_3,\ \cdots \text{ are nonnegative real numbers,} \\ a_1,\ a_2,\ a_3,\ \cdots \text{ are not all equal to } 0 \end{array} \Rightarrow \sum_1^\infty (a_1 a_2 a_3 \cdots a_n)^{1/n} < e \sum_1^\infty a_n.$$

By reading the first line of the proof, the agent sees that this intention of type 'to show' is then transformed into a further intention by including an additional hypothesis—the definition of the c_i sequence. This then yields the following subplan:

1. (S_1) **Show** $\displaystyle\sum_1^\infty (a_1 a_2 a_3 \cdots a_n)^{1/n} < e \sum_1^\infty a_n$, *from*
   - a_1, a_2, a_3, $\cdots$ are nonnegative real numbers,
   - a_1, a_2, a_3, $\cdots$ are not all equal to 0,
   - $c_1 c_2 c_3 \cdots c_n = (n+1)^n$ for $n = 1, 2, 3, \cdots$.

However, from the reader's perspective, adding this subplan violates $\mathsf{consistency}^*_{\mathsf{PA}}$. This is due to the fact that the reader has no reason to believe that adding the extra hypothesis—the definition of the c_i sequence—at this stage will result in a plan that can be successfully executed. More specifically, she has no reason to believe that adding this extra hypothesis will help establish the conclusion of S_1.[14]

We are now at stage 1 in figure 6.1. To get to stage 2, the reader sees from the proof that intention S_1 is itself transformed into further intentions through the generation of the following subplan:

1. (I_1) **Infer** $\displaystyle\sum_1^\infty (a_1 a_2 \cdots a_n)^{1/n} = \sum_1^\infty \frac{(a_1 c_1 a_2 c_2 \cdots a_n c_n)^{1/n}}{n+1}$, *from*
   - $c_1 c_2 c_3 \cdots c_n = (n+1)^n$ for $n = 1, 2, 3, \cdots$.

[14]It may be objected that the reader does have a reason to believe that introducing this extra hypothesis will help establish S_1, namely that this move is part of a successful proof. Clearly, this is not the right kind of reason that would satisfy the reader, for otherwise the reader's potential complaints imagined by Pólya would not make sense in the first place.

2. (S_2) **Show** $\sum_1^\infty (a_1a_2a_3\cdots a_n)^{1/n} < e\sum_1^\infty a_n$, *from*

- $a_1,\ a_2,\ a_3,\ \cdots$ are nonnegative real numbers,
- $a_1,\ a_2,\ a_3,\ \cdots$ are not all equal to 0,
- $c_1c_2c_3\cdots c_n = (n+1)^n$ for $n = 1,\ 2,\ 3,\ \cdots$,
- $\sum_1^\infty (a_1a_2\cdots a_n)^{1/n} = \sum_1^\infty \frac{(a_1c_1a_2c_2\cdots a_nc_n)^{1/n}}{n+1}$.

But here again we have another violation of $\mathsf{consistency}^*_{\mathsf{PA}}$ from the reader's perspective. This is due to the fact that, at this stage, the reader has no reason to believe that the plan that results from integrating this subplan can be successfully executed. More precisely, the reader has no reason to believe that the equality

$$\sum_1^\infty (a_1a_2\cdots a_n)^{1/n} = \sum_1^\infty \frac{(a_1c_1a_2c_2\cdots a_nc_n)^{1/n}}{n+1},$$

will help establish the conclusion of S_2.

In our framework, then, a typical reader working through Pólya's proof will find two $\mathsf{consistency}^*_{\mathsf{PA}}$ violations in the corresponding proof activity. We suggest that it is precisely these violations that underlie the objections Pólya noted against this proof. Recall, for example, that his final objection was that a reader will fail to grasp that the introduction of the c_i sequence "has some chances of success" [Pólya 1949, p. 685]. This objection directly reflects that the reader fails to believe that the plan which results from integrating the subplans discussed above can be successfully executed. The same is true for many other of Pólya's complaints. For example, a failure to "see any reason why" the approach would help the reader to prove the theorem, and being unable to follow the proof "with confidence" [Pólya 1949, p. 685] both support an interpretation in which the reader fails to believe that the plan she obtains can be successfully executed. In other words, Pólya's proof of Carleman's inequality is a concrete example of a proof which is judged to be defective precisely because, from the reader's perspective, its corresponding proof activity violates the norm of $\mathsf{consistency}^*_{\mathsf{PA}}$.[15]

7 Conclusion

Mathematical proofs in mathematical practice are produced by *rational* agents. In this respect, a mathematical proof is not a "simple juxtaposition" or an "haphazard collection" of deductive steps, that is, a mathematical proof is not a sequence of *arbitrary* deductive steps. Instead, a better characterization would be to say that:

A *mathematical proof* is a sequence of *rational* deductive steps.

In this paper, we have proposed an analysis of what it means for a deductive step in a mathematical proof to qualify as rational. We first observed that a mathematical proof is always the report of a successful proof activity, and that when we judge the rationality of a deductive step in a mathematical proof, we in fact judge the rationality of a deductive inference in a proof activity. We then identified the norms of rationality governing deductive inferences in proof activities, which we traced back to norms of *planning* rationality, and we provided evidence that these norms are indeed at play in mathematical practice. Taken together, this constitutes our proposed characterization of what Poincaré and Mac Lane might consider to be missing from the traditional view of mathematical proof.

Another way to formulate the point expressed by Poincaré and Mac Lane would be to say that the traditional view provides a too impoverished conception of the *structure* of mathematical proofs. Mac Lane put things in exactly this way when he wrote that: "proofs are not mere collections of atomic processes, but are rather complex combinations with a highly rational structure" [Mac Lane 1935, p. 130]. Following Mac Lane's terminology, we can say that a mathematical proof possesses

[15]There is reason to think that there is a whole class of proofs which, like Pólya's, are judged to be defective because, from a reader's perspective, their corresponding proof activities involve $\mathsf{consistency}^*_{\mathsf{PA}}$ violations. Such proofs are often referred to as *unmotivated* proofs. See Morris [forthcoming] for a detailed discussion of motivated and unmotivated proofs.

a *rational structure* in virtue of being composed of *rational* deductive steps. By offering a characterization of what is meant here by rational, we are thus providing an account of the rational structure of mathematical proofs. In short, what makes the structure of a proof *rational* is precisely that it is the direct product or outcome of a distinctive form of *rational planning agency*.[16]

The approach adopted here might be used to analyze the rational structure of pieces of mathematics other than proofs. In particular, it might help to analyze the rational structure or organization of whole mathematical developments. This is an issue that has recently attracted some philosophical attention, most notably from Sieg [2010] and Avigad [2020]. As in the case of mathematical proofs, one can conceive of whole mathematical developments as being the product of rational planning agents, thus opening the way to an analysis of their rational structure in terms of planning agency. This can be achieved by identifying the relevant mathematical activities in play, and by characterizing the form(s) of planning agency underlying them.

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[16] As a reviewer pointed out, one may wonder whether computer-generated proofs should also be credited with possessing a rational structure. Since computer-generated proofs are produced by computers which are themselves programmed by human agents, we may expect to find some traces of human rationality in their structures. Yet, these eventual traces will likely depend on the kind of computer-generated proofs under consideration. A proper treatment of this intriguing question would call for a dedicated study.

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